Risk Aversion Reversals*

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Abstract

We characterize when risk aversion reversals—situations in which a more risk-averse individual chooses a riskier asset more frequently—can arise in stochastic choice models. Reversals occur if and only if the ratio of the second derivatives of the individuals' utilities is unbounded. We show that reversals are robust to rich forms of noise—including non-Fechnerian, heterogeneous, and menu-dependent noise—and to utilities without constant risk aversion. Moreover, reversals are pervasive in traditional CARA- and CRRA-based models, even for empirically relevant risk coefficients and lottery choices, so these functional forms cannot be relied on to robustly model stochastic choice. We thus propose two parametric utility families that do not lead to reversals, offering well-behaved alternatives to CARA and CRRA for stochastic choice.

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1 Introduction

A precise understanding of how humans evaluate risk is central to economics. Since choices are inherently noisy, analysts deploy stochastic choice models to recover individuals' risk preferences and noise from choice data. The most widely used stochastic choice models for this purpose are *Fechnerian models*, a non-parametric class that includes the logit and probit models. In practice, the analyst chooses a functional form for utility and a functional form for noise which maps utility differences to choice probabilities (e.g., Harrison, List, and Towe, 2007; Von Gaudecker, Van Soest, and Wengström, 2011; Holzmeister and Stefan, 2021).

However, predictions from these models under CARA or CRRA—the most commonly used utility functions—suffer from a well-known problem. When two individuals differ in their risk-aversion parameters but face the same noise, the more risk-averse individual exhibits more risk-seeking behavior in some choices (Wilcox, 2008, 2011; Blavatskyy, 2011; Apesteguia and Ballester, 2018). In response to these critiques, subsequent work has argued that non-monotonicity is particularly worrisome under homoskedastic noise and has advocated allowing heteroskedasticity for calibration of noise to utility (Barseghyan, Molinari, O'Donoghue, and Teitelbaum, 2018; O'Donoghue and Somerville, 2024; Keffert and Schweizer, 2024). Moreover, it is common to model heterogeneity in both preferences and noise and to estimate them jointly (e.g., Hey and Orme, 1994; Von Gaudecker, Van Soest, and Wengström, 2011; Meissner, Gassmann, Faure, and Schleich, 2023).

In this paper, we show that the problems noted for Fechnerian models are far more pervasive than previously recognized: they persist even when individuals differ arbitrarily in their Fechnerian noise, when noise varies across menus, and when noise takes a more general, non-Fechnerian form. That is, under CARA and CRRA utilities, none of these generalizations can eliminate the counterintuitive predictions. We then characterize which utility functions, beyond CARA and CRRA, give rise to risk-aversion reversals and advocate for two parametric utility families that do not exhibit reversals under various noise specifications: a more risk-averse individual always chooses safer options more frequently. Beyond its intuitive appeal, this comparative static aligns with empirical findings (Bruner, 2017).

To build intuition, we begin with the baseline Fechnerian framework, in which each individual is associated with a Bernoulli utility function u and a strictly increasing function F. We refer to F as a Fechnerian noise structure, where

$$F(U(X) - U(Y))$$

is the probability that X is chosen over Y, and $U(X) = \mathbb{E}[u(X)]$. The most popular Fechnerian noise structures are CDFs of normal and logistic distributions, corresponding to probit and logit models. Indeed, any additive random utility models with i.i.d. shocks fit the Fechnerian framework.[1]

Our first result is that, for some pairwise choices between a safe and a risky lottery,[2] a more risk-averse individual chooses the risky option more often than a less risk-averse individual, even when the two differ arbitrarily in their Fechnerian noise structures (Theorem 1). Thus, the problems identified for the homoskedastic noise model cannot be resolved by jointly estimating risk and noise, even when noise is estimated non-parametrically.

For example, consider an analyst who observes Anne and Bob choosing from three distinct menus—each menu consists of a safe lottery—a sure payoff of \$8, \$10, or \$14—and a 50–50 risky lottery paying plus or minus \$4, \$6, or \$8 from the sure amount. The choice probabilities for the safe lotteries are shown in the second and third columns of Table 1. Assuming CRRA expected utilities for their risk preferences and normal distributions for their noise structures, Anne is estimated to be more risk averse and experience less noise than Bob. Indeed, the choice probabilities in Table 1 correspond to Anne having a CRRA coefficient of 0.8 and noise variance of 0.5, compared to Bob's 0.3 and 1, respectively.[3]

[1] Technically, for F to be strictly increasing, the difference in these shocks must have full support.

[2] We say that lottery X is safer than lottery Y if X dominates Y in the concave order. Equivalently, every expected utility maximizer with a concave utility function must prefer X to Y.

[3] I.e., the analyst deploys the probit model for Anne and Bob

$$F_A(U_A(X) - U_A(Y)) \quad \text{and} \quad F_B(U_B(X) - U_B(Y)),$$

respectively, where U_A and U_B are CRRA expected utility functions with relative risk aversions a and b, and F_A and F_B are the CDFs of normal distributions with zero means and respective variances σ_A^2 and σ_B^2.

Base Lottery	Probability of S		Probability of $3S$	
(S vs R)	Anne	Bob	Anne	Bob
8 vs $4, 12$	0.60	0.57	0.62	0.65
10 vs $4, 16$	0.65	0.62	0.69	0.74
14 vs $6, 22$	0.64	0.63	0.68	0.77

Table 1: CRRA Choice Probabilities for S with normal errors.

In the last two columns of Table 1, we report the choice probabilities of safe options whose outcomes are tripled from the base lotteries under the estimated parameters. Note that when the stakes are increased, Bob chooses safer options more frequently than Anne from all menus. Thus, despite Anne being estimated as more risk-averse and more precise, the model predicts that she will behave more risk-seeking than Bob once the stakes are scaled up.

This reversal pattern does not hinge on these specific CRRA risk coefficients and noise structures, nor on the specific choice of lotteries. Indeed, given lotteries X and Y with X safer than Y, we show how to construct lotteries X' and Y' offering higher potential rewards, such that X' remains safer than Y' yet leads to a reversal (Proposition 2). In the special case that the noise structures satisfy $F_A(t/\sigma_A) = F_B(t/\sigma_B)$ for some $\sigma_A, \sigma_B > 0$ (e.g., when both are probit, with different variances as the example above), a reversal occurs if both lotteries are scaled sufficiently: for all large enough scales k, Bob chooses kX over kY more frequently than Anne does (Proposition 1). For CARA utilities, reversals occur after shifting the lotteries by the same sufficiently large outcome under arbitrary noise structures for Anne and Bob (Proposition 2). Thus, CARA and CRRA Fechnerian models with arbitrary noise structures make the perverse and systematic prediction that those most inclined toward choosing safer options when stakes are small must become the least inclined when stakes are larger.

Assuming that this reversal pattern does not always bear out in reality—and indeed Bruner (2017) finds that it does not—our results indicate that the estimates yielded by these models will be sensitive to the level of stakes that are analyzed, making them unreliable.[4] Indeed, if the analyst instead observed the choice probabil-

[4]This echoes the critique of expected utility in Rabin (2000), which also reveals paradoxes when

ities of the scaled lotteries in Table 1, the analyst would estimate that Bob is more risk-tolerant than Anne, even if he exhibited more risk-averse behavior. As illustrated in that table, the stake increases are often modest, indicating that such reversals can emerge even in low-stakes experimental settings. Moreover, these models give counterintuitive, and perhaps counterfactual, out-of-sample predictions for choices made at larger or smaller stakes.

We next show that the paradoxical reversals of CARA and CRRA Fechnerian models are not artifacts of overly restrictive Fechnerian noise structures. To this end, we explore *simply scalable expected utility* (SSEU) models, which consist of a vNM utility function U and a noise structure $H\colon \mathbb{R}^2 \to [0,1]$ that translates the utilities of two alternatives into choice probabilities.[5] That is, $H(U(X), U(Y))$ is the probability that X is chosen over Y. SSEU models generalize Fechnerian models by not requiring choice probabilities to depend only on utility differences. We demonstrate that SSEU models based on CARA or CRRA expected utilities are guaranteed to produce reversals under a smoothness condition on noise structures (Theorem 2).

One might suspect, at this point, that these reversals arise because H is applied to vNM utilities, which are just representations of preferences and carry no cardinal significance. A natural alternative would be to calculate, for each lottery X, its certainty equivalent $u^{-1}(\mathbb{E}[u(X)])$, and apply H to these certainty equivalents. It turns out that this does not solve the problem. In fact, we will still have reversals, even if we more generally apply H to $f(\mathbb{E}[u(X)])$ for some strictly increasing f, because absorbing f into the noise structure H simply generates a new noise structure (Corollary 1).

We next examine whether allowing noise structures to vary across menus can eliminate these paradoxes. In a *menu-dependent Fechnerian expected-utility* (MFEU) model, each individual is associated with a vNM utility function and a *noise assignment* that specifies, for every pair of lotteries, a menu-specific Fechnerian noise structure. An important subclass of menu-dependent models arises in the additive random utility framework, when the random shocks to utility that are associated with

stakes are increased. While Rabin's result is about the curvature of utility functions in deterministic models, our result is about the interaction between noise structures and CARA/CRRA utilities.

[5] We require choice probabilities to be monotone in utility, meaning that H is increasing in its first argument and decreasing in its second. These models of noise were studied in depth by Tversky and Russo (1969).

each lottery are independent but not identically distributed. For example, lotteries with large stakes may induce increased attention and be associated with less-variable shocks. To rule out pathological noise assignments, we impose a continuity condition requiring that menus containing similar lotteries yield similar noise structures. Our next result shows that MFEU models necessarily yield reversals (Corollary 2). Thus, even when allowing for arbitrary noise, varying across individuals and menus, the CARA and CRRA expected utility functional forms lead to untenable predictions.

We next turn our attention toward the characterization of families of Bernoulli utilities that interact well with noise. Our final result is that when u_A is more risk averse than u_B, there exist reversals for every choice of noise structures if and only if the ratio

$$u_B''(x)/u_A''(x)$$

is unbounded (Theorem 3). Indeed, for CARA and CRRA utilities, this condition fails, leading to reversals in every model of noise we have considered. Moreover, this condition is not satisfied by a class of utility functions that generalizes CARA and CRRA (Corollary 3).

We conclude by advocating for two families of utility functions where there are no reversals under homoskedastic noise (see Section 6.1). These families are obtained by taking a linear combination of two distinct CARA (CRRA) utilities and are parameterized by the coefficient on the more risk-averse utility function. For example, for any $0 \leq r < s$, let crra_r and crra_s be CRRA Bernoulli utility functions with respective relative risk coefficients r and s. Summing $\text{crra}_r + a \cdot \text{crra}_s$ yields a family of utility functions parameterized by $a \geq 0$, with greater a corresponding to greater risk aversion. These simple families of utility functions offer well-behaved alternatives for empirical analysis. We discuss the properties of these utility families and the interpretation of the parameter a in Appendix I.

1.1 Related Literature

We contribute to a large body of literature on risky choices in the presence of noise (e.g., Becker, DeGroot, and Marschak, 1963; Harless and Camerer, 1994; Hey and Orme, 1994; Loomes, Moffatt, and Sugden, 2002; Blavatskyy, 2007). See Wilcox (2021) for a recent survey. In particular, we study Fechnerian noise structures and their generalizations. Axiomatic investigations of these models have been undertaken by Debreu (1958), Tversky and Russo (1969), and Fudenberg, Iijima, and Strzalecki (2015), among others. See Strzalecki (2025) for an extensive review.

The papers most closely related to ours are Wilcox (2011) and Apesteguia and Ballester (2018), which demonstrate that CARA and CRRA utilities coupled with identical Fechnerian noise are problematic: a more risk-averse individual exhibits more risk-seeking behavior in some pairwise choices.[6] Our work builds on their insights in several ways. First, we allow for heterogeneous noise structures and show that the paradox persists without any parametric assumptions on how noise differs across individuals (Theorem 1). Second, we show that the reversals occur under more general non-Fechnerian noise (Theorem 2).

We further allow noise to vary across menus, recognizing that some comparisons may involve greater cognitive effort or uncertainty. Menu-dependent noise in risky choice has been extensively studied. Hey (1995), Buschena and Zilberman (2000), and Loomes (2005) model Fechnerian noise that depends on, for example, value differences between lotteries and question difficulty. More recently, He and Natenzon (2024) and Shubatt and Yang (2024) characterize Fechnerian noise that depends on a measure of distance between alternatives. We show that, under very general conditions, the paradox persists even when we allow noise to vary across menus (Corollary 2).

Finally, we characterize which families of utility functions, beyond CARA and CRRA, suffer from these reversals (Theorem 3) and propose alternative parametric families of utilities that do not suffer from the problem. This is in contrast with the solutions proposed by Wilcox (2011) and Apesteguia and Ballester (2018) who retain CARA and CRRA utilities.

Wilcox (2011) considers a model of menu-dependent noise and shows it is monotone for lotteries over three fixed outcomes. Nevertheless, Apesteguia and Ballester (2018) show that this model is no longer monotone for more than three outcomes and propose instead the random parameter model, where each individual is associated with a distribution over CARA or CRRA preferences and chooses probabilistically as if they draw a random preference from their distribution. Random parameter models do not give rise to reversals, but they are based on a different foundation

[6] Wilcox (2011) shows that other parameterizations of CARA and CRRA utilities also lead to reversals. This corresponds to two individuals having noise structures that are related by a scale factor. We study this special case in Proposition 1.

than traditional models of noise. In particular, in random parameter models there is no longer a core preference over alternatives such that an individual usually chooses the more preferred outcome. This is in contrast with our proposal to use alternative vNM utility functions which maintains the idea that individuals choose according to a core preference under noise.

Another paper that studies risk aversion in stochastic choice is Blavatskyy (2011), which examines when one individual is always more likely to choose a sure thing over a lottery than another individual. A main result of that paper is that this notion of stochastic comparative risk aversion is too strong to be compatible with any Fechnerian expected utility models.

We compare the riskiness of lotteries using the concave order of Rothschild and Stiglitz (1970). In the deterministic setting, Kihlstrom, Romer, and Williams (1981) and Ross (1981) have noted that insurance and risk premia are not increasing in the Arrow-Pratt order. Ross (1981) thus proposes a stronger notion of comparative risk preferences than Arrow-Pratt that leads to monotonicity. Our characterization result (Theorem 3) shows that this stronger notion implies our condition, and thus is sufficient to eliminate reversals under some noise structures.

Finally, we contribute to a literature examining failures of expected utility theory in predicting choices across varying ranges of stakes. The most prominent critique in this area is by Rabin (2000), who demonstrates that within the expected utility framework, even modest levels of risk aversion exhibited over small-stakes, imply absurd, counterfactual levels of risk aversion for large stakes. While this result holds for deterministic choice, Propositions 1 and 2 echo Rabin (2000) in the stochastic setting: out of sample predictions made from small-stakes observations are counterintuitive.

2 Preliminaries

We denote by $\mathcal{L}$ the set of all bounded real random variables defined over a non-atomic probability space $(\Omega, \Sigma, \mathbb{P})$. We use the term *lotteries* to refer to elements of $\mathcal{L}$. A stochastic choice rule $\rho\colon \mathcal{L} \times \mathcal{L} \to \mathbb{R}$, where $\rho(X, Y) + \rho(Y, X) = 1$ and $\rho(X, Y)$ is the probability of choosing X over Y. A vNM *utility function* is a map $U\colon \mathcal{L} \to \mathbb{R}$ given by $U(X) = \mathbb{E}[u(X)]$ for some strictly increasing, concave, and continuous Bernoulli utility $u\colon \mathbb{R} \to \mathbb{R}$.

We consider an individual that evaluates lotteries according to a vNM utility U but faces noise and chooses stochastically according to ρ. We think of ρ as a noisy expression of the underlying utility U. For example, an individual with utility U whose noise takes a logit form has choice probabilities

$$\rho(X,Y) = \frac{e^{\beta U(X)}}{e^{\beta U(X)} + e^{\beta U(Y)}},$$

for some $\beta > 0$. When β is fixed across all menus, ρ belongs to the class of Fechnerian models, where choice probabilities are a function of utility differences. We discuss Fechnerian models in depth in Section 3. More generally, the logit scale parameter β may depend on (X,Y), in which case ρ is a menu-dependent Fechnerian model. We study these general models and models where choice probabilities are a general function of utilities in Sections 4 and 5.

We say that X dominates Y in the *concave order*, denoted $X \geq_c Y$ if $\mathbb{E}[g(X)] \geq \mathbb{E}[g(Y)]$ for all concave functions $g\colon \mathbb{R} \to \mathbb{R}$. We denote the strict part of $\geq_c$ by $>_c$. Recall that $X \geq_c Y$ if and only if Y is a mean-preserving spread of X. If $X \geq_c Y$, then $U(X) \geq U(Y)$ for all vNM utilities U.[7] We will often use the variables S and R when referring to lotteries where $S >_c R$, in order to highlight that S is *safer* while R is *riskier*.

2.1 Stochastic Comparative Risk Aversion

We are interested in comparing the choices of individual A with those of B. Each individual i is associated with a utility function U_i and stochastic choice model ρ_i. We will often refer to agents and their associated stochastic choice models interchangeably. We say A is *more risk-averse than* B if $U_A(X) \geq U_A(y)$ implies $U_B(X) \geq U_B(y)$ for any $X \in \mathcal{L}$ and degenerate lottery y.[8]

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[7]Since we defined vNM utilities as expectations of concave Bernoulli utilities, they are increasing in $\geq_c$.

[8]This is Yaari (1969)'s notion of comparative risk, which coincides with the Arrow-Pratt notion of comparative risk since U_A and U_B are vNM utility functions. I.e., U_A being more risk-averse than U_B is equivalent to the concavity of $u_A \circ {u_B}^{-1}$, when u_A and u_B are the corresponding Bernoulli utility functions to U_A and U_B. Indeed, CARA and CRRA utilities are totally ordered by this order.

Definition 1. *When A is more risk-averse than B, we say that there is a* reversal *if there are lotteries $S >_{\mathrm{c}} R$, where*

$$\rho_A(S,R) < \rho_B(S,R).$$

Moreover, we say that S and R generate a reversal.

This means that A, the more risk-averse individual, exhibits more risk-tolerant behavior by choosing the safer lottery S less frequently than B. A's behavior can also be seen as more risk-tolerant in the following sense: Let s_A and s_B denote A's and B's respective choice probabilities of S. The ex-ante lottery that A receives is the compound lottery resulting in S with probability s_A and in R with probability $1 - s_A$, and likewise for B. Since $s_A < s_B$, A's compound lottery is riskier than B's, so A evidently makes riskier choices than B.

We show that many models of noise yield reversals under the Constant Absolute Risk Aversion (CARA) and Constant Relative Risk Aversion (CRRA) utility specifications, which are the most commonly used parametric families of utility functions in both theory and practice. CARA and CRRA utilities are defined by $\mathrm{CARA}_a(X) = \mathbb{E}[\mathrm{cara}_a(X)]$ and $\mathrm{CRRA}_a(X) = \mathbb{E}[\mathrm{crra}_a(X)]$, where where $a \geq 0$ is the coefficient of absolute/relative risk aversion and

$$\mathrm{cara}_a(x) = \begin{cases} x & a = 0 \\ \frac{1-\mathrm{e}^{-ax}}{a} & a > 0, \end{cases} \qquad \mathrm{crra}_a(x) = \begin{cases} \frac{x^{1-a}-1}{1-a} & a \neq 1 \\ \ln(x) & a = 1. \end{cases}$$

The coefficients reflect varying levels of risk aversion, with higher coefficients corresponding to greater aversion to risk.[9] To ensure that $\mathrm{CRRA}_a(X)$ is well defined, we restrict our attention to X that is non-negative and bounded away from 0 whenever referring to CRRA.

3 Fechnerian Models

We first consider models of the form

$$\rho(X,Y) = F(U(X) - U(Y)),$$

[9] Note that there are many possible ways of parameterizing CARA and CRRA preferences, since applying a positive affine transformation to each Bernoulli utility does not change the underlying preference/coefficient. Nevertheless, we will show that our results hold for every possible parameterization of these families.

where $F\colon \mathbb{R} \to [0,1]$ is a strictly increasing function that is continuously differentiable with non-vanishing derivative. Moreover, $F(t) + F(-t) = 1$. We denote by $\mathcal{F}$ the set of all such functions and refer to a model of the form (U, F), where U is a vNM utility and $F \in \mathcal{F}$, as a *Fechnerian expected utility* (FEU) model.[10] In these models, the larger the difference between $U(X)$ and $U(Y)$, the higher the probability that X is chosen.

FEU models are among the most widely used stochastic choice models (Becker, DeGroot, and Marschak, 1963; Loomes, Moffatt, and Sugden, 2002). An important subclass of FEU models yields choice probabilities

$$\mathbb{P}(U(X) + \varepsilon \geq U(Y) + \varepsilon') = F(U(X) - U(Y)), \tag{1}$$

where ε and ε' are i.i.d. continuous random variables and F is the CDF of $\varepsilon - \varepsilon'$. This subclass includes many widely used discrete choice models such as the logit and probit models, where the random shocks are Gumbel and Gaussian, respectively.

In this section, we focus on FEU models and show that the CARA and CRRA specifications display a paradox: they predict that a more risk-averse individual will sometimes choose a riskier asset more frequently than a more risk-tolerant individual under any Fechnerian noise specifications.

Theorem 1. *For any $F_A, F_B \in \mathcal{F}$ and distinct* CARA (CRRA) *utilities U_A, U_B, there exist reversals.*

Theorem 1 is an impossibility result for the commonly used CARA/CRRA-expected-utility Fechnerian models since reversals arise under arbitrary pairs of noise structures and risk coefficients. In the case of $F_A = F_B$, i.e., when individuals face the same noise structures, it was shown by Wilcox (2011) and Apesteguia and Ballester (2018) that the probability of choosing the safer option from a pair is non-monotone in the CARA/CRRA coefficients.[11] In practice, however, it is common to model heterogeneity in both preferences and noise and to estimate them jointly (Hey and Orme, 1994; Von Gaudecker et al., 2011). Indeed, subsequent work has argued

[10]The assumptions that F must be continuously differentiable with positive derivative can be relaxed without affecting our results, provided these assumptions hold at 0. Note that we do not assume that $\lim_{t\to\infty} F(t) = 1$, nor that $\lim_{t\to-\infty} F(t) = 0$, although this will be the case for the examples we study.

[11]When $F_A = F_B$, Theorem 1 follows from Proposition 3 of Apesteguia and Ballester (2018).

that non-monotonicity is particularly worrisome under homoskedastic noise and has advocated allowing heteroskedasticity for calibration of noise to utility (Barseghyan et al., 2018; O'Donoghue and Somerville, 2024; Keffert and Schweizer, 2024).

Yet, perhaps surprisingly, Theorem 1 demonstrates that non-monotonicity persists even when noise structures vary arbitrarily across individuals. Thus, no matter how we jointly estimate risk coefficients and noise structures for two individuals, the resulting estimated models (CARA_a, F_A) and (CARA_b, F_B) will predict reversals. Because reversals are inherent to these models, we must be cautious when using these estimates to rank individuals by risk aversion.

Note that choice probabilities are not invariant to reparameterizations of CARA and CRRA utilities. For example, consider the model (U, F), where U is a CARA or CRRA utility and $F \in \mathcal{F}$. Let $V(X) = \frac{1}{c}U(X) + d$ for some $c > 0$. While V represents the same preference as U, replacing U with V has the same effect on choice probabilities as changing the noise structure from F to G which is defined by $G(t) = F(t/c)$. However, since G is also a member of $\mathcal{F}$, it follows from Theorem 1 that reparameterizations of the CARA or CRRA utility family cannot resolve the non-monotonicity.

3.1 Scale-Family Heteroskedasticity

To develop intuition for Theorem 1, we first consider individuals with distinct CARA or CRRA utilities U_A and U_B with respective coefficients a and b, where $a > b$ and whose noise structures are related by a scale factor. That is, $F_A(t) = F(t/\sigma_A)$ and $F_B(t) = F(t/\sigma_B)$ for some $\sigma_A, \sigma_B > 0$ and common $F \in \mathcal{F}$.[12] The choice probabilities of S over R are then given by

$$F\left(\frac{U_A(S) - U_A(R)}{\sigma_A}\right) \tag{2}$$

$$F\left(\frac{U_B(S) - U_B(R)}{\sigma_B}\right). \tag{3}$$

Since F is strictly increasing, the pair (S, R) generates a reversal (i.e., (3) is strictly larger than (2)) if and only if

$$\frac{U_B(S) - U_B(R)}{U_A(S) - U_A(R)} > \frac{\sigma_B}{\sigma_A}. \tag{4}$$

[12] This is the case, for example, when F_A and F_B are both normal or both logistic CDFs.

Note that $\frac{\sigma_B}{\sigma_A}$, which is the relative noise level, does not depend on the lotteries. Given this, the next lemma immediately implies that to generate a reversal, we can start with any $S >_c R$, and get a reversal by adding to both a large enough constant x (in the CARA case) or multiplying both by a large enough constant (in the CRRA case).

Lemma 1. *For any $S >_c R$, $a, b > 0$, and all $x \in \mathbb{R}$ and $k > 0$,*

1. $\dfrac{\mathrm{CARA}_b(S+x) - \mathrm{CARA}_b(R+x)}{\mathrm{CARA}_a(S+x) - \mathrm{CARA}_a(R+x)} = C_1 \cdot \mathrm{e}^{(a-b)x}$

2. $\dfrac{\mathrm{CRRA}_b(k \cdot S) - \mathrm{CRRA}_b(k \cdot R)}{\mathrm{CRRA}_a(k \cdot S) - \mathrm{CRRA}_a(k \cdot R)} = C_2 \cdot k^{a-b}$,

for some positive constants C_1 and C_2.

Proof of Lemma 1. Let $S >_c R$ and $a, b > 0$. For CARA, we have

$$\begin{aligned}\frac{\mathrm{CARA}_b(S+x) - \mathrm{CARA}_b(R+x)}{\mathrm{CARA}_a(S+x) - \mathrm{CARA}_a(R+x)} &= \frac{a}{b} \frac{\mathbb{E}\left[\mathrm{e}^{-b(R+x)}\right] - \mathbb{E}\left[\mathrm{e}^{-b(S+x)}\right]}{\mathbb{E}[\mathrm{e}^{-a(R+x)}] - \mathbb{E}[\mathrm{e}^{-a(S+x)}]} \\ &= \frac{a}{b}\left(\frac{\mathbb{E}\left[\mathrm{e}^{-bR}\right] - \mathbb{E}\left[\mathrm{e}^{-bS}\right]}{\mathbb{E}[\mathrm{e}^{-aR}] - \mathbb{E}[\mathrm{e}^{-aS}]}\right)\mathrm{e}^{(a-b)x}.\end{aligned}$$

For CRRA, we have

$$\begin{aligned}\frac{\mathrm{CRRA}_b(k \cdot S) - \mathrm{CRRA}_b(k \cdot R)}{\mathrm{CRRA}_a(k \cdot S) - \mathrm{CRRA}_a(k \cdot R)} &= \frac{1-a}{1-b}\left(\frac{\mathbb{E}\left[S^{1-b}\right] - \mathbb{E}\left[R^{1-b}\right]}{\mathbb{E}[S^{1-a}] - \mathbb{E}[R^{1-a}]}\right)\frac{k^{1-b}}{k^{1-a}} \\ &= \frac{1-a}{1-b}\left(\frac{\mathbb{E}\left[S^{1-b}\right] - \mathbb{E}\left[R^{1-b}\right]}{\mathbb{E}[S^{1-a}] - \mathbb{E}[R^{1-a}]}\right)k^{a-b}.\end{aligned}$$

Since positive risk coefficients a and b for CARA/CRRA utilities correspond to risk-averse utilities,

$$C_1 = \frac{a}{b}\left(\frac{\mathbb{E}\left[\mathrm{e}^{-bR}\right] - \mathbb{E}\left[\mathrm{e}^{-bS}\right]}{\mathbb{E}[\mathrm{e}^{-aR}] - \mathbb{E}[\mathrm{e}^{-aS}]}\right), \quad C_2 = \frac{1-a}{1-b}\left(\frac{\mathbb{E}\left[S^{1-b}\right] - \mathbb{E}\left[R^{1-b}\right]}{\mathbb{E}[S^{1-a}] - \mathbb{E}[R^{1-a}]}\right) \tag{5}$$

are positive. □

Note that when $a > b$, the ratios are strictly increasing in x and k and tend to infinity. Moreover, these properties hold for any $S >_c R$. We thus have the following result which strengthens the existence of reversal pairs established in Theorem 1 when noise structures are related by a scale factor.

Proposition 1. *Let $\sigma_A, \sigma_B > 0$ and $F \in \mathcal{F}$. Let $F_A(t) = F(\frac{t}{\sigma_A})$, $F_B(t) = F(\frac{t}{\sigma_B})$, and $S >_{\mathrm{c}} R$. Let U_A, U_B be distinct CARA utilities and let V_A, V_B be distinct CRRA utilities. Then there exist unique x_0 and k_0 such that*

1. *$S + x$ and $R + x$ generate a reversal for (U_A, F_A) and (U_B, F_B) if and only if $x > x_0$;*

2. *kS and kR generate a reversal for (V_A, F_A) and (V_B, F_B) if and only if $k > k_0$.*

Note that the critical values x_0 and k_0 given in Proposition 1 depend on the noise scale parameters σ_A, σ_B, risk coefficients, and the baseline lotteries. The values can be derived from (4) and the positive constants C_1 and C_2 given in (5). I.e., critical values x_0 and k_0 solve for $C_1 \cdot \mathrm{e}^{(a-b)x} = \frac{\sigma_B}{\sigma_A}$ and $C_2 \cdot k^{(a-b)} = \frac{\sigma_B}{\sigma_A}$.[13] For example, when the CRRA coefficients a and b are 0.8 and 0.3, respectively, and the noise variances are $\sigma_A = 0.5$ and $\sigma_B = 1$, the critical values that generate reversals for lotteries given in Table 1 are all less than 3, which illustrates that reversals often occur under empirically relevant risk coefficients and lotteries.

Proposition 1 highlights two problems with the CARA and CRRA Fechnerian models. First, there is a disconnect between the standard Arrow-Pratt notion of comparative risk-aversion developed in the deterministic framework with the resulting stochastic choice predictions. Namely, more risk-averse individuals are not more likely to make risk-averse choices. This counterintuitive prediction is contradicted by empirical evidence that risk aversion is negatively correlated with decision noise, as measured by the frequency with which individuals choose the concave-order dominated option (Bruner, 2017). In terms of estimation, this means that even if Bob chooses safer lotteries more frequently than Anne, we may conclude that Bob is more risk-tolerant than Anne.[14] This is the case, for example, if we only observe the choice probabilities of the scaled lotteries in Table 1.

Second, these models make the systematic prediction that those most inclined to choose safer options when stakes are small must become the least inclined once the

[13]Plugging C_1 and C_2 into the equations, we obtain

$$x_0 = \frac{1}{a-b} \ln \left(\frac{\sigma_B}{\sigma_A} \frac{b}{a} \left(\frac{\mathbb{E}\left[\mathrm{e}^{-aR}\right] - \mathbb{E}\left[\mathrm{e}^{-aS}\right]}{\mathbb{E}[\mathrm{e}^{-bR}] - \mathbb{E}[\mathrm{e}^{-bS}]} \right) \right), \quad k_0 = \left(\frac{\sigma_B}{\sigma_A} \frac{1-b}{1-a} \left(\frac{\mathbb{E}\left[S^{1-a}\right] - \mathbb{E}\left[R^{1-a}\right]}{\mathbb{E}[S^{1-b}] - \mathbb{E}[R^{1-b}]} \right) \right)^{\frac{1}{a-b}}.$$

[14]This holds under any consistent estimator.

stakes grow modestly larger. Even without a specific model of risk aversion in mind, this prediction is implausible.

3.2 Arbitrary Fechnerian Noise Structures

So far we have considered the special case that F_A and F_B are related by a scale factor. To prove the more general case considered in Theorem 1, we establish the following lemma, which shows that the properties of CARA and CRRA utilities given in Lemma 1 give rise to non-monotonicity for any $F_A, F_B \in \mathcal{F}$.

Lemma 2. *Let $F_A, F_B \in \mathcal{F}$. Let U_A and U_B be vNM utility functions with U_A more risk averse. Suppose that for each $M \in \mathbb{R}$ there exist lotteries $S >_{\mathrm{c}} R$ such that*

$$\frac{U_B(S) - U_B(R)}{U_A(S) - U_A(R)} \geq M. \tag{6}$$

Then there are reversals for (U_A, F_A) and (U_B, F_B).

Lemma 2 identifies a sufficient condition on the vNM utility functions to yield the negativity result in Theorem 1. By Lemma 1, CARA and CRRA utilities satisfy this condition, and thus Theorem 1 follows immediately from these lemmas.

Let $S >_{\mathrm{c}} R$ satisfying (6) for some $M \in \mathbb{R}$. The key step of the proof of Lemma 2 is to construct $S' >_{\mathrm{c}} R'$ such that

$$\frac{U_B(S) - U_B(R)}{U_A(S) - U_A(R)} = \frac{U_B(S') - U_B(R')}{U_A(S') - U_A(R')},$$

while making the utility differences arbitrarily small. To this end, we consider the menu $(S, R_\lambda S)$, where $\lambda \in (0,1)$ and $R_\lambda S$ is distributed as a compound lottery that yields R with probability λ and S with probability $1-\lambda$. As $\lambda \to 0$, the utility difference between S and $R_\lambda S$ vanishes. Lemma 2 exploits the linearity of vNM utilities, which ensures that the ratio of utility differences is independent of λ. It also relies on the differentiability of F_A and F_B to approximate them around 0 by affine functions. Lemma 2 is proved in Appendix A.

Note that by Lemma 1, (6) is satisfied for all concave-ordered lotteries under sufficient shifting/scaling. Moreover, for CARA, the utility difference $U(S+x) - U(R+x)$ tends to 0 as x tends to infinity (see Lemma 4 in Appendix B). For CRRA, on the other hand, $U(kS) - U(kR)$ increases with k for relative risk coefficients less than unity. We therefore apply an additional transformation of scaling down the probability of receiving the risky prospect so that the utility difference is arbitrarily small. Based on these observations, the following proposition extends the non-monotonicity result of Proposition 1 to all $F_A, F_B \in \mathcal{F}$.

Proposition 2. *Let $F_A, F_B \in \mathcal{F}$ and $S >_{\mathrm{c}} R$. Let U_A and U_B be distinct CARA utilities and V_A and V_B be distinct CRRA utilities. Then there exist x_0, $k_0 > 0$ such that*

1. *$S + x$ and $R + x$ generate a reversal for (U_A, F_A) and (U_B, F_B) for $x > x_0$.*
2. *$k \cdot S$ and $k \cdot (R_\lambda S)$ generate a reversal for (V_A, F_A) and (V_B, F_B) for $k > k_0$ and λ small enough.*

Proposition 2 is proved in Appendix B. While CARA preferences are invariant to changes in background wealth, the above proposition shows that non-monotonicity occurs in FEU models under CARA utilities for any concave-ordered lotteries under sufficient background wealth. Likewise, while CRRA preferences are invariant to the scaling of stakes and to mixing with the safe lottery, these transformations can always generate a non-monotonicity.

Proposition 2 demonstrates that non-monotonicity is a pervasive problem. It holds for all pairs of risk parameters, for all pairs of Fechnerian noise specifications, and for all concave-ordered lotteries after a transformation that preserves the concave ordering as well as the family of preferences being measured.

4 Simply Scalable Models

We next consider models of the form

$$\rho(X, Y) = H(U(X), U(Y)),$$

where $H \colon \mathbb{R}^2 \to [0, 1]$ is increasing in the first argument and decreasing in the second argument. To ensure that the sum of choice probabilities is 1, we require that $H(s, t) + H(t, s) = 1$. We refer to H as a *noise structure*. When H is differentiable, we denote by H_i the partial derivative of H with respect to the i^{th} argument. We denote by $\mathcal{H}$ the set of all continuously differentiable noise structures H such that

$H_1 > 0$ and $H_2 < 0$.[15] We refer to a model of the form (U, H) where U is a vNM utility and $H \in \mathcal{H}$, as a *simply scalable expected utility* (SSEU) model.

Notably, choice probabilities resulting from an SSEU model (U, H) are monotone in the utility U. I.e., if $U(X) \geq U(Y)$, then X is chosen over any alternative Z more frequently than Y is chosen over Z.[16] Using a strengthening of this property, Tversky and Russo (1969) characterize a more general model where the choice probability of X over Y is given by some noise structure H and some arbitrary utility function V.

We think of H as a measure of preference intensities between alternatives that depends on their utilities. When this measure is simply the utility difference, this reduces to FEU. SSEU models are much more flexible, allowing for any measure of intensities that is monotone in the utilities. A notable example of an SSEU model is the strict utility model of Luce and Suppes (1965) in which the choice probabilities are determined by the utility ratio, rather than by the utility difference.[17]

The next result shows that the paradoxes of FEU models with CARA or CRRA utilities are not artifacts of overly restrictive Fechnerian noise structures.

Theorem 2. *For any $H_A, H_B \in \mathcal{H}$ and any distinct* CARA (CRRA) *utilities U_A and U_B, there are reversals.*

One may guess, at this point, that reversals arise in SSEU models under CARA or CRRA utilities because H is applied to vNM utility U, which is merely a representation of the preference and carries no cardinal significance. A natural alternative would be to calculate, for each lottery X, its certainty equivalent $u_A^{-1}(\mathbb{E}[u_A(X)])$,

[15] This implies that H is strictly increasing in the first argument and strictly decreasing in the second argument, and furthermore that these partial derivatives do not vanish.

[16] In Appendix H, we show that in any stochastic choice model with this property, the choice probability of X over Y takes the form

$$H(U(X), U(Y)),$$

for some H that is increasing in the first argument and satisfying $H(s, t) + H(t, s) = 1$. This is weaker than the requirement that $H \in \mathcal{H}$, which additionally requires continuous non-vanishing partial derivatives.

[17] The strict utility model yields choice probabilities

$$H(U(X), U(Y)) = \frac{U(X)}{U(X) + U(Y)}.$$

and apply H to the certainty equivalents. The next result shows that this does not solve the problem. In fact, we will still have reversals, even if we more generally apply H to $f \circ U$ where f is any continuously differentiable function with positive derivative. This is because H_f defined by $H_f(s,t) = H(f(s), f(t))$ is also a member of $\mathcal{H}$.

Corollary 1. *Let f_A and f_B be continuously differentiable functions with positive derivatives. Then, for any $H_A, H_B \in \mathcal{H}$ and any distinct* CARA (CRRA) *utilities U_A and U_B, there are reversals for $(f_A \circ U_A, H_A)$ and $(f_B \circ U_B, H_B)$.*

To see what drives non-monotonicity in these models, note that when a noise structure H is applied to a utility function, it measures the strength of preference between alternatives in a way that depends on the particular utility representation. CARA and CRRA expected utilities exhibit pathological comparative statics in their risk coefficients, preventing any hope of monotonicity. When these utilities are transformed to, say, certainty equivalents, the continuous differentiability of the monotone transformation f ensures that the resulting utilities change locally like expected utilities. It turns out that this remnant of CARA/CRRA expected utility is enough to preclude monotonicity.

Theorem 2 is proved in Appendix E. The proof is based on the observation that the class of SSEU models is contained within the broader class of menu-dependent Fechnerian models, which also suffer from non-monotonicity. We discuss this class of models and show how they generalize SSEU in the upcoming section.

5 Menu-Dependent Models with Arbitrary vNM Utilities

In the previous sections, we studied models in which noise structures vary across individuals but not across menus for the same individual. In this section, we study a more flexible model that incorporates menu-dependent noise structures. For example, consider a probit model whose variance depends on the variances of the lotteries in the menu. This can capture the intuitive idea that more variable lotteries may generate more noise, allowing for mistake probabilities to remain high even when stakes are high.

In a *menu-dependent Fechnerian expected-utility* (MFEU) model, each individual is associated with a vNM utility function U and menu-dependent Fechnerian noise structures, i.e., a map $\Phi\colon \mathcal{L}\times\mathcal{L}\to\mathcal{F}$ and

$$\Phi_{X,Y}(U(X)-U(Y))$$

is the probability that they choose X over Y, where $\Phi_{X,Y}$ is short for $\Phi(X,Y)$. We refer to Φ as a *noise assignment*, and require that $\Phi_{X,Y}(t)+\Phi_{Y,X}(-t)=1$.[18]

An example of MFEU models arises from replacing the identicality assumption on the shock terms in (1) with a symmetry requirement. That is, the choice probability of X over Y is

$$\mathbb{P}(U(X)+\varepsilon_X \geq U(Y)+\varepsilon_Y)=\Phi_{X,Y}(U(X)-U(Y)),$$

where ε_X and ε_Y are independent and symmetric about zero (but not necessarily identically distributed) random variables and $\Phi_{X,Y}$ is the CDF of $\varepsilon_X-\varepsilon_Y$.[19]

Although MFEU models involve Fechnerian noise structures, the flexibility of allowing noise structures to vary by menu is enough for MFEU to generate the same stochastic choice predictions as any SSEU model with a general noise structure $H\in\mathcal{H}$. In fact, under a fixed utility U, the choice probabilities from MFEU models are equivalent to those of a menu-dependent version of SSEU. Indeed, if the choice probability of X over Y is given by $\Psi_{X,Y}(U(X),U(Y))$ for some function $\Psi\colon\mathcal{L}\times\mathcal{L}\to\mathcal{H}$, then the noise assignment $\Phi\colon\mathcal{L}\times\mathcal{L}\to\mathcal{F}$ defined by

$$\Phi_{X,Y}(t)=\Psi_{X,Y}\left(\frac{U(X)+U(Y)}{2}+\frac{t}{2},\frac{U(X)+U(Y)}{2}-\frac{t}{2}\right) \tag{7}$$

generates the same choice probabilities. Note, however, that two menu-independent SSEU models with the same noise specifications $H_A=H_B$ will correspond to different MFEU noise assignments Φ^A and Φ^B when $U_A\neq U_B$, since the assignment (7) depends on the utility.

One way in which MFEU generalizes the menu-independent SSEU model of Section 4 is that, unlike in SSEU where choice probabilities can only depend on utilities, choice probabilities from MFEU models may depend on the distributions of the lotteries. In fact, they may even depend on the joint distribution of the lotteries in a menu, since Φ is a function of the random variables in $\mathcal{L}$. For example, X may

[18]Since $\Phi_{X,Y},\Phi_{Y,X}\in\mathcal{F}$, this requirement implies $\Phi_{X,Y}=\Phi_{Y,X}$.

[19]The symmetry assumption on the shocks ensures that $\Phi_{X,Y}\in\mathcal{F}$. We can relax this assumption by only requiring that the difference is symmetric.

be chosen over Y more frequently if X dominates Y state-wise rather than in terms of first-order stochastic dominance.[20]

Without any restrictions on the noise assignment Φ, the only implication of MFEU models is that individuals will choose their preferred lottery more than half of the time.[21] In order to rule out pathological noise assignments, we require that menus with similar lotteries are associated with similar noise structures. Formally, we define topologies on $\mathcal{L}$ and on $\mathcal{F}$ and require that the noise assignment be continuous. We say that a sequence of lotteries $X_1, X_2, \ldots$ converges to a lottery X if for all increasing and continuous functions u it holds that

$$\mathbb{E}[|u(X_n) - u(X)|] \to 0.$$

In other words, $X_1, X_2, \ldots$ converges to X if the sequence of random utilities $u(X_1), u(X_2), \ldots$ converges to $u(X)$ in L^1 for every increasing and continuous Bernoulli u. An important property of this topology is that vNM utilities are continuous. Moreover, noise assignments are not required to depend only on the marginal distributions of the lotteries in the menu; they may, for example, depend on their joint distribution.

We associate each noise structure in $\mathcal{F}$ with its derivatives and say that Φ is *continuous* if whenever $X_n \to X$ and $Y_n \to Y$, it holds that the derivatives of Φ_{X_n,Y_n} converge compactly to that of $\Phi_{X,Y}$.[22]

It turns out that many noise assignments are continuous. In particular, our topology on $\mathcal{L}$ is very fine (finer than any L^p topology for $1 \leq p < \infty$), making many functions on $\mathcal{L}$ continuous. Moreover, compact convergence is not too strict a requirement in many natural applications. For example, for normal and logistic distributions, parameter convergence implies compact convergence of densities. E.g., a sequence of densities ϕ_n of normal distribution with means $\mu_n \to \mu$ and variances

[20]This distinction can only be determined from the joint distribution of (X, Y), not from their marginals.

[21]Since $\Phi_{X,Y} \in \mathcal{F}$, if $U(X) - U(Y) \geq 0$, then $\Phi_{X,Y}(U(X) - U(Y)) \geq \frac{1}{2}$ with equality if and only if $U(X) = U(Y)$. This requirement is weaker than the central requirement of SSEU, namely that choice probabilities must be monotone in the utility. See Appendix H.

[22]That is, on every compact set $K \subset \mathbb{R}$,

$$\lim_n \left(\sup_{t \in K} \left| \frac{d}{dt} \Phi_{X_n,Y_n}(t) - \frac{d}{dt} \Phi_{X,Y}(t) \right| \right) = 0.$$

$\sigma_n \to \sigma$ compactly converges to ϕ with corresponding mean μ and variance σ. In fact, parameter convergence is sufficient for the location-scale family of distributions with continuous densities (see Appendix C). Thus, a *weak-utility model* in which the choice probability of X over Y is given by

$$F\left(\frac{U(X)-U(Y)}{\sigma(X,Y)}\right), \tag{8}$$

where $F \in \mathcal{F}$ and $\sigma\colon \mathcal{L}\times\mathcal{L} \to (0,\infty)$ is symmetric, is a continuous MFEU model if σ is continuous.

5.1 Characterization

Our next theorem characterizes when two vNM utilities necessarily lead to reversals under any continuous MFEU noise structures. Given two increasing and concave Bernoulli utility functions u_A and u_B, we say that u_A is *absolutely more concave than* u_B, and write $u_A \geq_{\text{abs}} u_B$ if $u_A - u_B$ is concave. Moreover, we say that u_A is *relatively more concave* than u_B, and write $u_A \geq_{\text{rel}} u_B$ if there exists $k > 0$ such that $k \cdot u_A(x) - u_B(x)$ is concave.[23] We extend these notions to vNM utilities $U_A = \mathbb{E}[u_A]$ and $U_B = \mathbb{E}[u_B]$, writing $U_A \geq_{\text{abs}} U_B$ if $u_A \geq_{\text{abs}} u_B$ and $U_A \geq_{\text{rel}} U_B$ if $u_A \geq_{\text{rel}} u_B$. For twice differentiable functions with $u_A'' < 0$, the condition that $u_A \geq_{\text{rel}} u_B$ is equivalent to the boundedness of $u_B''(x)/u_A''(x)$, i.e., that $u_B'' = O(u_A'')$ on $\mathbb{R}$.

Theorem 3. *Let U_A and U_B be vNM utilities with U_A more risk averse. There exist continuous noise assignments Φ^A and Φ^B such that there are no reversals if and only if $U_A \geq_{\text{rel}} U_B$.*

The necessity of $U_A \geq_{\text{rel}} U_B$ to avoid reversals is proved in Appendix D. A key idea is to show that if U_A is not relatively more concave than U_B, then the ratio

$$\frac{U_B(S)-U_B(R)}{U_A(S)-U_A(R)}$$

is unbounded over the set of lottery pairs $S \geq_{\text{c}} R$. Under continuity of the noise assignment, we show that this unboundedness leads to reversals using an argument

[23]Recall that u_A is more risk averse than u_B in the Arrow-Pratt sense if $u_A = f \circ u_B$ for some increasing and concave function f. This notion of comparative concavity is logically independent of both $\geq_{\text{abs}}$ and $\geq_{\text{rel}}$. In Appendix F, we show that the stronger comparative risk notion of Ross (1981) implies $\geq_{\text{rel}}$ and discuss an interpretation of his stronger notion in the stochastic framework.

similar to that of Theorem 1. This argument relies on the fact that continuity imposes a bound on the relative noisiness of A and B for some lottery comparisons. We note that limitations on relative noisiness leads to reversals even under discontinuous noise assignments. For example, in the weak utility model (8), if $\sigma_B(S,R)/\sigma_A(S,R)$ is bounded, there will necessarily be reversals even if σ_A and σ_B are not continuous.

We prove sufficiency of $U_A \geq_{\text{rel}} U_B$ here by showing that even under a simple heteroskedastic probit model, without menu dependence, this condition suffices. Indeed, suppose that $ku_A - u_B$ is concave for some $k > 0$. Let Φ^B assign to every menu the CDF of a standard normal distribution, denoted by F, and let Φ^A always assign the CDF of a normal distribution with zero mean and standard deviation $1/k$. For any lotteries $S \geq_{\text{c}} R$,

$$\Phi^A_{S,R}(U_A(S) - U_A(R)) = F(k(U_A(S) - U_A(R))) \tag{9}$$

$$\Phi^B_{S,R}(U_B(S) - U_B(R)) = F(U_B(S) - U_B(R)). \tag{10}$$

Since $ku_A - u_B$ is concave, $kU_A(S) - U_B(S) \geq kU_A(R) - U_B(R)$. Hence, since F is strictly increasing, (9) exceeds (10), meaning that there are no reversals.

Theorem 3 establishes a simple test for identifying whether a pair of vNM utilities generates reversals. Indeed, for CARA and CRRA utilities, $\text{cara}''_b(x)/\text{cara}''_a(x) = (b/a)e^{-(b-a)x}$ and $\text{crra}''_b(x)/\text{crra}''_a(x) = (b/a)x^{a-b}$ are unbounded for $a > b$. Thus, as a corollary, the paradoxical property of simply scalable CARA and CRRA models cannot be resolved by allowing noise structures that depend continuously on the menu.

Corollary 2. *For any continuous noise assignments and distinct* CARA (CRRA) *utilities there are reversals.*

Corollary 2 highlights that the non-monotonicity result of the menu-independent Fechnerian models (Theorem 1) is not merely a byproduct of asymptotic properties of choice probabilities when stakes are increased. Indeed, in the following example noise assignments are engineered so that choice probabilities are scale-invariant, yet monotonicity does not obtain.

Example 1. *Let $a > b > 0$, $U_A = \text{CRRA}_a$, and $U_B = \text{CRRA}_b$. Let F be the CDF of the standard normal distribution. Define the noise assignments Φ^A and Φ^B by*

$$\Phi^A_{X,Y}(t) = F\left(\frac{t}{\sigma_A(X,Y)}\right) \qquad \Phi^B_{X,Y}(t) = F\left(\frac{t}{\sigma_B(X,Y)}\right),$$

where

$$\sigma_A(X,Y) = \mathbb{E}\Big[X^{a-1} + Y^{a-1}\Big]^{-1}, \qquad \sigma_B(X,Y) = \mathbb{E}\Big[X^{b-1} + Y^{b-1}\Big]^{-1}.$$

Then for all $k > 0$, we have

$$\Phi^A_{kX,kY}(U_A(kX) - U_A(kY)) = \Phi^A_{X,Y}(U_A(X) - U_A(Y))$$
$$\Phi^B_{kX,kY}(U_B(kX) - U_B(kY)) = \Phi^B_{X,Y}(U_B(X) - U_B(Y)).$$

In this example, the noise structure is given by the CDF of normal distributions with mean 0 and standard deviation $\sigma_A(X,Y)$. When $a \geq 1$, σ_A decreases as X and Y are scaled up, capturing the behavioral property that individuals pay more attention when facing higher stakes. Moreover, the standard deviations are chosen so that each individual's probability of choosing βX over βY does not depend on $\beta > 0$, as $\sigma_A(\beta X, \beta Y) = \beta^{1-a}\sigma_A(X,Y)$ and

$$U_A(\beta X) - U_A(\beta Y) = \beta^{1-a}(U_A(X) - U_A(Y)).$$

Since $\sigma_A(X_n, Y_n) \to \sigma_A(X,Y)$ whenever $X_n \to X$ and $Y_n \to Y$, Φ^A is continuous (see Proposition 5 in Appendix C). Thus, it follows from Corollary 2 that even these calibrated models cannot avoid the paradox.

Corollary 2 demonstrates that CARA and CRRA utilities generate reversals whenever choices are noisy, even if we allow noise structures to vary across individuals and menus. Thus, if we hope to model the stochastic choice behavior of A and B, where A consistently chooses safer lotteries more frequently than B, we must forgo CARA and CRRA utilities. Moreover, the boundedness of u''_B/u''_A is violated by many parametric families of utilities that are used to model risk aversion. In particular, we show in Appendix G that expo-power utility—a popular generalization of the CARA/CRRA families—violates the condition, leading to reversals under any continuous noise assignments. We take this on in the next section, where we suggest alternative vNM utilities that can accommodate consistent comparative statics under standard models of noise.

6 Monotone Alternatives to CARA and CRRA

In the previous section, we showed that when two vNM utilities have an unbounded ratio of second derivatives, there are necessarily reversals under very general noise structures. Conversely, when the ratio of second derivatives is bounded, there exist heteroskedastic noise structures such that there are no reversals. In this section, we demonstrate when two vNM utilities will not lead to reversals under any homoskedastic Fechnerian models and, more generally, under homoskedastic MFEU models. We provide two parametric families of utilities such that any two utilities within a family satisfies this condition. We then discuss the economic properties of these utility families.

Recall that $u_A \geq_{\text{abs}} u_B$ if $u_A - u_B$ is concave. The next proposition states that this absolute comparative concavity notion leads to no reversals under identical Fechnerian noise.

Proposition 3. *Let $F \in \mathcal{F}$ and let U_A more risk-averse than U_B. Then there are no reversals under (U_A, F) and (U_B, F) if and only if $U_A \geq_{\text{abs}} U_B$.*

Proof of Proposition 3. Since Fechnerian noise structures $F \in \mathcal{F}$ are strictly increasing, the choice probability of S is higher for A than B if and only if the utility difference is higher for A than B, i.e.,

$$U_A(S) - U_A(R) \geq U_B(S) - U_B(R),$$

or equivalently,

$$U_A(S) - U_B(S) \geq U_A(R) - U_B(R). \tag{11}$$

We now show that this inequality holds for all lotteries $S \geq_{\text{c}} R$ if and only if $U_A \geq_{\text{abs}} U_B$.

Let $h = u_A - u_B$. If h is concave, then (11) is satisfied by the definition of the concave order. Conversely, for the sake of contradiction, suppose that h is not concave. Then there are $x, y \in \mathbb{R}$ and $\lambda \in (0, 1)$ such that

$$h(\lambda x + (1 - \lambda)y) < \lambda h(x) + (1 - \lambda)h(y).$$

Let S denote a degenerate lottery that always pays $\lambda x + (1 - \lambda)y$, and let R denote a lottery that pays x with probability λ and y with complementary probability. Then

$$U_A(S) - U_B(S) = h(\lambda x + (1 - \lambda)y) < \lambda h(x) + (1 - \lambda)h(y) = U_A(R) - U_B(R),$$

contradicting (11). Thus $u_A - u_B$ is concave, meaning $U_A \geq_{\text{abs}} U_B$. □

In the case where A and B have different noise structures, i.e., $F_A \neq F_B$, a sufficient condition for no reversals is if $U_A \geq_{\text{abs}} U_B$ and F_A is *more precise than* F_B, in the sense that $F_A(t) \geq F_B(t)$ for $t \geq 0$. A more precise noise structure translates the same utility difference into a higher probability of choosing a more preferred alternative. Since A is risk averse, increasing A's precision will only increase the probability that A chooses S. In particular,

$$F_A(U_A(S) - U_A(R)) \geq F_A(U_B(S) - U_B(R)) \geq F_B(U_B(S) - U_B(R)),$$

where the first inequality follows from Proposition 3. Of course, A may experience much more noise than B, in which case A will make more mistakes and may choose riskier options more frequently than B, even with $U_A \geq_{\text{abs}} U_B$.

The same conclusion holds when A and B face the same menu-dependent Fechnerian noise assignments $\Phi^A = \Phi^B$ or when Φ^A assigns a more precise Fechnerian noise than Φ^B to each menu. Note, however, that we cannot conclude that there are no reversals under simply scalable expected utility models where A and B face the same noise structure $H \in \mathcal{H}$ and $U_A \geq_{\text{abs}} U_B$ since Φ^A and Φ^B that give rise to H depend on U_A and U_B, which are not identical.

6.1 Parametric Families of Utility Functions

We conclude by proposing parametric utility families ordered by both risk aversion and $\geq_{\text{abs}}$, leading to monotone comparative statics under identical (menu-dependent) Fechnerian noise. Recall that for strictly increasing concave functions f and g, f is *more concave* than g if $f \circ g^{-1}$ is increasing and concave. The following proposition provides a simple way to construct a parametric utility family that yields consistent comparative statics.

Proposition 4. *Let f and g be strictly increasing concave functions such that f is more concave than g. Let $u_A(x) = af(x) + g(x)$ and $u_B(x) = bf(x) + g(x)$, where $a > b \geq 0$. Then $U_A = \mathbb{E}[u_A]$ is more risk-averse than $U_B = \mathbb{E}[u_B]$ and $U_A \geq_{\text{abs}} U_B$.*

These utilities are parametrized by the coefficient on the more risk-averse Bernoulli utility function f, i.e., the higher the coefficient, the greater the risk aversion.[24]

[24]Indeed,

$$-\frac{u_A''(x)}{u_A'(x)} \geq -\frac{u_B''(x)}{u_B'(x)} \iff (a-b)\left(-\frac{f''(x)}{f'(x)} + \frac{g''(x)}{g'(x)}\right) \geq 0.$$

Moreover, $u_A - u_B = (a-b)f$ is concave since f is concave, so $u_A \geq_{\text{abs}} u_B$, and increasing, since f is increasing and $a > b$. Thus, monotonicity of choice probabilities in the risk coefficient also holds for all lotteries comparable by second-order stochastic dominance (SOSD). We write $X \geq_{\text{ssd}} Y$ if X dominates Y in the SOSD order.

Remark 2. *Let f and g be strictly increasing concave functions such that f is more concave than g. Let $u_A(x) = af(x) + g(x)$ and $u_B(x) = bf(x) + g(x)$, where $a > b \geq 0$. Then for any $F \in \mathcal{F}$,*

$$F(U_A(X) - U_A(Y)) \geq F(U_B(X) - U_B(Y))$$

for all $X \geq_{\text{ssd}} Y$.

When f and g are CARA (CRRA) utilities with different coefficients, u_A and u_B correspond to sum-ex (sum-power) utilities (see, e.g., Farquhar and Nakamura, 1987; Bell, 1988; Pedersen and Satchell, 2001). Proposition 4 thus suggests a replacement of the CARA utility family or CRRA utility family for estimation. One specifies an upper and lower bound on absolute risk aversion, corresponding to two CARA utilities, and then estimates the weight on each utility. The same exercise can be done with bounds on relative risk aversion and CRRA utilities. In Appendix I, we discuss properties of sum-ex and sum-power utility functions and how to interpret the parameters.

7 Conclusion

This paper establishes that the paradox long noted for CARA and CRRA utilities under homoskedastic Fechnerian noise is not an artifact of restrictive parametric assumptions, but instead reflects a deeper incompatibility between these utility forms and noisy choice. We show that the same non-monotone comparative statics arise under highly flexible noise specifications—including heterogeneous noise across individuals and menu-dependent noise.

We obtain a simple characterization of whether two utility functions can yield sensible predictions in the presence of noise. This condition rules out even the more

Since f is more concave than g, it follows that $a \geq b$ is equivalent to U_A being more risk-averse than U_B.

general expo-power utilities, and points to new parametric families that restore intuitive comparative statics. These families, which include sum-ex and sum-power utilities, provide empirically tractable, well-behaved alternatives for measuring risk preferences in noisy environments. We leave to future work an empirical examination of how risk aversion and decision noise co-vary across different stake levels, as well as systematic tests of these proposed utility families under a range of noise specifications.

We conclude by noting that the continuity requirement on noise assignments may be restrictive in some settings. For example, discontinuous noise assignments may capture interesting features of binary comparisons, such as salience or complexity. Understanding which discontinuous models generate reversals thus remains an interesting direction for future research.

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A Proof of Theorem 1

Lemma 1 provides sequences of lottery pairs (S_n, R_n) along which the ratio of expected-utility differences $(U_B(S_n) - U_B(R_n))/(U_A(S_n) - U_A(R_n))$ diverges, for CARA and CRRA utilities. Lemma 2 establishes that when such a sequence exists there will be reversals. An important subtlety of Lemma 2 is that the reversal-generating lotteries may not be part of this sequence. Indeed, Lemma 2 relies on the following lemma, which establishes that under the additional condition that utility differences tend to zero, there will be lotteries in the sequence that generate reversals.

Lemma 3. *Let $F_A, F_B \in \mathcal{F}$ and let U_A and U_B be risk-averse vNM utility functions with U_A more risk averse. There exist $M \in \mathbb{R}$ and $\delta > 0$ such that all lotteries $S >_c R$ satisfying*

$$\frac{U_B(S) - U_B(R)}{U_A(S) - U_A(R)} \geq M \quad \textit{and} \quad U_i(S) - U_i(R) < \delta$$

for $i = A, B$ generate a reversal.

Proof of Lemma 3. Let $f_A(t) = \frac{d}{dt}F_A(t)$ and $f_B(t) = \frac{d}{dt}F_B(t)$. Since F_A is continuously differentiable, f_A is continuous at 0 so that for each $\varepsilon_A > f_A(0)$ there is $\delta_A > 0$ such that for each $t \in (0, \delta_A)$, $f_A(t) < \varepsilon_A$ and $F_A(t) < \frac{1}{2} + t\varepsilon_A$. Since F_B is continuously differentiable and $f_B > 0$, for each $0 < \varepsilon_B < f_B(0)$, there exists $\delta_B > 0$ such that for any $t \in (0, \delta_B)$, $f_B(t) > \varepsilon_B$ and $F_B(t) > \frac{1}{2} + t\varepsilon_B$. Let $\delta = \min\{\delta_A, \delta_B\}$ and let $M = \varepsilon_A/\varepsilon_B$.

Let S and R as in the statement of the lemma. Then

$$\begin{aligned} F_B(U_B(S) - U_B(R)) &> \frac{1}{2} + (U_B(S) - U_B(R))\varepsilon_B \\ &\geq \frac{1}{2} + (U_A(S) - U_A(R))\varepsilon_A \\ &> F_A(U_A(S) - U_A(R)). \end{aligned}$$

□

In light of Lemma 3, Lemma 2 follows from the observation that for any lotteries $S >_{\mathrm{c}} R$ and $\varepsilon > 0$, there exist lotteries $S' >_{\mathrm{c}} R'$ with the same utility difference ratio such that the each utility difference is less than ε.

Proof of Lemma 2. Let $F_A, F_B \in \mathcal{F}$ and let U_A and U_B be risk-averse vNM utility functions with U_A more risk averse. By Lemma 3, there exist $M \in \mathbb{R}$ and $\delta > 0$ such that all lotteries $S >_{\mathrm{c}} R$ satisfying

$$\frac{U_B(S) - U_B(R)}{U_A(S) - U_A(R)} \geq M \quad \text{and} \quad U_i(S) - U_i(R) < \delta$$

for $i = A, B$ generate a reversal. By hypothesis, there exist lotteries $S >_{\mathrm{c}} R$ satisfying the first inequality.

For $\lambda \in (0, 1)$, let $R_\lambda S$ denote a lottery distributed as a compound lottery that yields R with probability λ and yields S with probability $1 - \lambda$. Then $S >_{\mathrm{c}} R_\lambda S$ and

$$\frac{U_B(S) - U_B(R_\lambda S)}{U_A(S) - U_A(R_\lambda S)} = \frac{U_B(S) - U_B(R)}{U_A(S) - U_A(R)},$$

by the linearity of expected utility, i.e., $U(R_\lambda S) = \lambda U(R) + (1 - \lambda)U(S)$ for any vNM utility U. Moreover, as λ tends to zero, $U_i(S) - U_i(R_\lambda S)$ for $i = A, B$ tend to zero. Hence, S and $R_\lambda S$ generate a reversal for all λ small enough.

□

B Proof of Proposition 2

Proposition 2 shows how to construct the reversal-generating lotteries of Theorem 1 by generalizing Proposition 1 to arbitrarily different noise structures. The CARA case relies on the following lemma about diminishing utility differences as background wealth increases.

Lemma 4. *Let $S \geq_{\mathrm{c}} R$. Then*

$$\lim_{x\to\infty} \mathrm{CARA}_a(S+x) - \mathrm{CARA}_a(R+x) = 0$$

for all $a > 0$.

Proof. Since S and R have equal means, we suppose, without loss of generality, that $\mathbb{E}[S] = \mathbb{E}[R] = 0$. Let $\mathrm{cara}_a(x) = \frac{1-\mathrm{e}^{-ax}}{a}$ denote the CARA Bernoulli utility under coefficient a, and let m denote the essential infimum of S.[25] By monotonicity, $\mathrm{cara}_a(x) - \mathrm{cara}_a(m+x) \geq \mathrm{cara}_a(x) - \mathrm{CARA}_a(S+x)$ and by concavity of u_a, since $\mathbb{E}[S] = 0$, $\mathrm{cara}_a(x) - \mathrm{CARA}_a(S+x) \geq 0$. Moreover, since $m \leq 0$ and cara_a is concave,

$$\frac{d}{dx}\mathrm{cara}_a(m+x) \cdot |m| \geq \mathrm{cara}_a(x) - \mathrm{cara}_a(m+x).$$

Note that $\frac{d}{dx}\mathrm{cara}_a(m+x) = \mathrm{e}^{-a(m+x)}$. Thus

$$\mathrm{e}^{-a(m+x)} \cdot |m| \geq \mathrm{cara}_a(x) - \mathrm{cara}_a(m+x) \geq \mathrm{cara}_a(x) - \mathrm{CARA}_a(S+x) \geq 0.$$

Taking the limit as $x \to \infty$, we see that $\mathrm{cara}_a(x) - \mathrm{CARA}_a(S+x) \to 0$. Since S was an arbitrary mean-zero lottery and $\mathbb{E}[R] = 0$, we have

$$\lim_{x\to\infty} \mathrm{cara}_a(x) - \mathrm{CARA}_a(R+x) = 0$$

as well, concluding the proof. □

Proof of Proposition 2. For the case of CARA utilities, Lemma 1 establishes that for any pair of lotteries $S >_{\mathrm{c}} R$, the ratio

$$\frac{U_B(S+x) - U_B(R+x)}{U_A(S+x) - U_A(R+x)}$$

[25] This is the largest value that S exceeds with probability 1.

tends to infinity with x, while Lemma 4 establishes that the numerator and denominator go to zero. Hence, the result follows from Lemma 3.

For the CRRA case, Lemma 1 establishes that for any pair of lotteries $S >_{\mathrm{c}} R$, the ratio

$$\frac{U_B(k \cdot S) - U_B(k \cdot R)}{U_A(k \cdot S) - U_A(k \cdot R)}$$

tends to infinity with k. As in the proof of Lemma 2,

$$\frac{U_B(k \cdot S) - U_B(k \cdot R_\lambda S)}{U_A(k \cdot S) - U_A(k \cdot R_\lambda S)} = \frac{U_B(k \cdot S) - U_B(k \cdot R)}{U_A(k \cdot S) - U_A(k \cdot R)},$$

and as λ tends to zero, $U_i(k \cdot S) - U_i(k \cdot R_\lambda S)$ tends to zero for $i = A, B$, and the result follows from Lemma 3. □

C Compact Convergence of Location-Scale Families

In this appendix, we show that for location-scale families, parameter convergence suffices for compact convergence.

Let $f : \mathbb{R} \to [0, \infty)$ be a continuous probability density function. Then $\mu \in \mathbb{R}$ and $\sigma > 0$ parameterize the location-scale family with densities

$$f_{\mu,\sigma}(t) = \frac{1}{\sigma} f\left(\frac{t-\mu}{\sigma}\right).$$

Proposition 5. *If $(\mu_n, \sigma_n) \to (\mu, \sigma)$, then $(f_{\mu_n,\sigma_n})_n$ converges compactly to $f_{\mu,\sigma}$.*

Proof of Proposition 5. Let $K \subset \mathbb{R}$ be compact.

$$\begin{aligned}
&\sup_{t\in K} |f_{\mu_n,\sigma_n}(t) - f_{\mu,\sigma}(t)| \\
&\quad = \sup_{t\in K} \left| \frac{1}{\sigma_n} f\left(\frac{t-\mu_n}{\sigma_n}\right) - \frac{1}{\sigma} f\left(\frac{t-\mu}{\sigma}\right) \right| \\
&\quad = \sup_{t\in K} \left| \frac{1}{\sigma_n} f\left(\frac{t-\mu_n}{\sigma_n}\right) - \frac{1}{\sigma} f\left(\frac{t-\mu_n}{\sigma_n}\right) + \frac{1}{\sigma} f\left(\frac{t-\mu_n}{\sigma_n}\right) - \frac{1}{\sigma} f\left(\frac{t-\mu}{\sigma}\right) \right| \\
&\quad \leq \left| \frac{1}{\sigma_n} - \frac{1}{\sigma} \right| \sup_{t\in K} \left| f\left(\frac{t-\mu_n}{\sigma_n}\right) \right| + \frac{1}{\sigma} \sup_{t\in K} \left| f\left(\frac{t-\mu_n}{\sigma_n}\right) - f\left(\frac{t-\mu}{\sigma}\right) \right|
\end{aligned}$$

Note that since $\sigma_n \to \sigma$ and $\mu_n \to \mu$, there is a compact set S such that $\frac{t-\mu_n}{\sigma_n} \in S$ for all n high enough and $t \in K$. Since f is bounded on S, the first term tends to 0, and since $\frac{t-\mu_n}{\sigma_n}$ converges to $\frac{t-\mu}{\sigma}$ uniformly on K, and f is uniformly continuous on S, the second term tends to 0 as well. □

D Proof of Theorem 3

Theorem 3 provides a characterization of vNM utilities that lead to consistent predictions under some noise structures. The proof of Theorem 3 makes use of Proposition 3 and the following lemma, which is a strengthening of Lemma 2.

Lemma 5. *Let $\Phi^A, \Phi^B \colon \mathcal{L} \times \mathcal{L} \to \mathcal{F}$ be continuous noise assignments. Let U_A and U_B be vNM utility functions with U_A more risk averse. Suppose that for each $M \in \mathbb{R}$ there exist lotteries $S >_{\mathrm{c}} R$ such that*

$$\frac{U_B(S) - U_B(R)}{U_A(S) - U_A(R)} \geq M.$$

Then there are reversals for (U_A, Φ^A) and (U_B, Φ^B).

Proof of Lemma 5. Let U_A and U_B be as in the statement of the lemma. Let $Z, Z', \Lambda \in \mathcal{L}$ such that Z and Z' have the same distribution, and Λ is uniformly distributed on $[0,1]$ and independent of Z and Z'. Let $F^A = \Phi^A(Z, Z')$ and $F^B = \Phi^B(Z, Z')$ and let $f^A(t) = \frac{d}{dt}F^A(t)$ and $f^B(t) = \frac{d}{dt}F^B(t)$. Let $\varepsilon_A > f^A(0)$ and $\varepsilon_B < f^B(0)$. Since f^A and f^B are positive and continuous at 0, there is $\delta > 0$ such that for all $0 < t < \delta$, $f^A(t) < \varepsilon_A$ and $f^B(t) > \varepsilon_B$.

By assumption, there are lotteries $X >_{\mathrm{c}} Y$ such that

$$\frac{U_B(X) - U_B(Y)}{U_A(X) - U_A(Y)} > \frac{\varepsilon_A}{\varepsilon_B}.$$

Since the inequality only depends on the distributions of X and Y, we may choose X and Y to be independent of Λ. We define the random variables S_λ and R_λ by

$$S_\lambda(\omega) = \begin{cases} X(\omega) & \Lambda(\omega) \leq \lambda \\ Z(\omega) & \Lambda(\omega) > \lambda \end{cases}$$

and

$$R_\lambda(\omega) = \begin{cases} Y(\omega) & \Lambda(\omega) \leq \lambda \\ Z'(\omega) & \Lambda(\omega) > \lambda. \end{cases}$$

Note that for $\lambda \in (0,1)$, $S_\lambda >_{\mathrm{c}} R_\lambda$ and

$$U_A(S_\lambda) - U_A(R_\lambda) = \lambda(U_A(X) - U_A(Y)),$$

since $U_A(Z) = U_A(Z')$. Hence,

$$\frac{U_B(S_\lambda) - U_B(R_\lambda)}{U_A(S_\lambda) - U_A(R_\lambda)} = \frac{U_B(X) - U_B(Y)}{U_A(X) - U_A(Y)} > \frac{\varepsilon_A}{\varepsilon_B}, \tag{12}$$

and $U_A(S_\lambda) - U_A(R_\lambda), U_B(S_\lambda) - U_B(R_\lambda) > 0$.

Let $F^A_\lambda = \Phi^A(S_\lambda, R_\lambda)$, $F^B_\lambda = \Phi^B(S_\lambda, R_\lambda)$ and let $f^A_\lambda(t) = \frac{d}{dt}F^A_\lambda(t)$ and $f^B_\lambda(t) = \frac{d}{dt}F^B_\lambda(t)$. Since $S_\lambda \to Z$ and $R_\lambda \to Z'$ as $\lambda \to 0$, continuity of Φ^A and Φ^B imply that $f^A_\lambda \to f^A$ and $f^B_\lambda \to f^B$ compactly. Thus, there is $\delta' > 0$ and λ_A such that for all $\lambda \in (0, \lambda_A)$ and all $t \in [0, \delta']$, $f^A_\lambda(t) < \varepsilon_A$ and $F^A_\lambda(t) < \frac{1}{2} + t\varepsilon_A$. Likewise, there is λ_B such that for all $\lambda \in (0, \lambda_B)$ and all $t \in [0, \delta']$, $F^B_\lambda(t) > \frac{1}{2} + t\varepsilon_B$.

For λ small enough,

$$U_A(S_\lambda) - U_A(R_\lambda) < \delta' \quad \text{and} \quad U_B(S_\lambda) - U_B(R_\lambda) < \delta'.$$

Thus, for λ small enough, we have

$$\begin{aligned} F^B_\lambda(U_B(S_\lambda) - U_B(R_\lambda)) &> \frac{1}{2} + (U_B(S_\lambda) - U_B(R_\lambda))\varepsilon_B \\ &> \frac{1}{2} + (U_A(S_\lambda) - U_A(R_\lambda))\varepsilon_A \\ &> F^A_\lambda(U_A(S_\lambda) - U_A(R_\lambda)). \end{aligned}$$

The second inequality follows from (12).

□

Proof of Theorem 3. Let U_A be more risk averse than U_B. In Section 5, we showed that if $U_A \geq_{\text{rel}} U_B$ then there are continuous (indeed, constant) noise assignments Φ^A, Φ^B such that there are no reversals under (U_A, Φ^A) and (U_B, Φ^B). We now show that when there will always be reversals if U_A is not relatively more concave than U_B. Indeed, suppose that, for all $k > 0$, it is not the case that $kU_A - U_B$ is concave, i.e., kU_A is not absolutely more concave than U_B. Then, by Proposition 3, for each $k > 0$, there are lotteries $S >_{\text{c}} R$ such that

$$k(U_A(S) - U_A(R)) < U_B(S) - U_B(R).$$

Hence, the result follows from Lemma 5. □

E SSEU Models

Recall that in an SSEU model, the choice probability of X over Y is given by $H(U(X), U(Y))$ for some noise structure $H\colon \mathbb{R}^2 \to [0,1]$ where H is increasing in the first argument and decreasing in the second argument and $H(s,t) + H(t,s) = 1$. Also recall that $\mathcal{H}$ is the set of all continuously differentiable noise structures H whose partial derivatives are nowhere zero.

Proof of Theorem 2. First, we show that SSEU models are a subclass of menu-dependent Fechnerian models. Indeed, let $H \in \mathcal{H}$ and let U be a CARA or CRRA vNM utility. Define $\Phi_{X,Y}(t) = H(\bar{U} + \frac{t}{2}, \bar{U} - \frac{t}{2})$ where $\bar{U} = \frac{U(X)+U(Y)}{2}$. Then

$$\Phi_{X,Y}(U(X) - U(Y)) = H(U(X), U(Y)).$$

Note that $\Phi_{X,Y}(t) + \Phi_{X,Y}(-t) = 1$ since $H(s,t) + H(t,s) = 1$. Let $\phi_{X,Y}(t) = \frac{d}{dt}\Phi_{X,Y}(t)$. Then

$$2\phi_{X,Y}(t) = H_1\left(\bar{U} + \frac{t}{2}, \bar{U} - \frac{t}{2}\right) - H_2\left(\bar{U} + \frac{t}{2}, \bar{U} - \frac{t}{2}\right).$$

Since H_1 is continuous and positive and H_2 is continuous and negative, it follows that $\phi_{X,Y}(t)$ is continuous and positive. Thus, $\Phi_{X,Y} \in \mathcal{F}$ for all $(X,Y) \in \mathcal{L} \times \mathcal{L}$.

It remains to be shown that Φ is a continuous assignment, i.e., when $X_n \to X$ and $Y_n \to Y$, it holds that $\phi_{X_n,Y_n} \to \phi_{X,Y}$ uniformly on each compact set. Letting $\bar{U}_n = \frac{U(X_n)+U(Y_n)}{2}$, we have

$$2\phi_{X_n,Y_n}(t) = H_1\left(\bar{U}_n + \frac{t}{2}, \bar{U}_n - \frac{t}{2}\right) - H_2\left(\bar{U}_n + \frac{t}{2}, \bar{U}_n - \frac{t}{2}\right).$$

Since U is a vNM utility function, it is continuous, and $\bar{U}_n \to \bar{U}$. Since H_1 and H_2 are continuous, they are uniformly continuous on compact sets. Thus, for any compact $K \subset \mathbb{R}$,

$$\lim_n \sup_{t\in K} |\phi_{X_n,Y_n}(t) - \phi_{X,Y}(t)| = 0.$$

The result follows from Corollary 2. □

F Comparison with Standard Comparative Risk Notions

Note that U_A is more risk-averse than U_B in the Arrow-Pratt sense if and only if, for all x,

$$\frac{u''_B(x)}{u''_A(x)} \leq \frac{u'_B(x)}{u'_A(x)}.$$

While stochastic comparative risk depends on bounding the ratio of second derivatives by a constant, the traditional Arrow-Pratt notion of comparative risk depends on bounding this ratio by the ratio of marginal utilities. Thus, when $u'_B(x)/u'_A(x)$ is bounded, the traditional notion of comparative risk is sufficient for the stochastic notion. On the other hand, Ross' (1981) stronger comparative notion requires

$$\frac{u''_B(x)}{u''_A(x)} \leq k \leq \frac{u'_B(x)}{u'_A(x)}, \tag{13}$$

for some $k > 0$ and all x, implying the boundedness of u''_B/u''_A, i.e., $u_A \geq_{\text{rel}} u_B$. Moreover, Ross (1981) shows (13) is equivalent to the existence of k such that $ku_A - u_B$ is concave and decreasing, while $u_A \geq_{\text{rel}} u_B$ does not require $ku_A - u_B$ to be decreasing.

The next proposition states that, in the presence of noise related by a scale factor, Ross' (1981) notion captures stochastic comparative risk aversion stemming from preferences rather than greater choice precision. To formalize this, we introduce the following notion of relative choice precision.

We say that A has *greater choice precision* than B if, whenever X first-order dominates Y, A chooses X more than B.

Proposition 6. *Let $U_A = \mathbb{E}[u_A]$ and $U_B = \mathbb{E}[u_B]$ be vNM utilities with U_A more risk averse than U_B. Define $F_A(t) = F(t/\sigma_A)$ and $F_B(t) = F(t/\sigma_B)$ for some $\sigma_A, \sigma_B > 0$ and $F \in \mathcal{F}$. Then there are no reversals and B has greater choice precision if and only if $u_A/\sigma_A - u_B/\sigma_B$ is decreasing and concave.*

Proof of Proposition 6. For any $X, Y \in \mathcal{L}$, A chooses X over Y more than B if and only if

$$U_A(X)/\sigma_A - U_B(X)/\sigma_B \geq U_A(Y)/\sigma_A - U_B(Y)/\sigma_B. \tag{14}$$

Thus, if $u_A/\sigma_A - u_B/\sigma_B$ is decreasing and concave, then A chooses X less often when X first-order dominates Y and more often when X concave-order dominates Y.

Conversely, suppose that there are no reversals and B has greater choice precision. It follows from Proposition 3 that $u_A/\sigma_A - u_B/\sigma_B$ is concave. Moreover, for any $x > y$, a degenerate lottery X paying x first-order dominates Y paying y. Since B has greater choice precision, the right-hand side of (14) is larger, and we conclude that $u_A/\sigma_A - u_B/\sigma_B$ is decreasing.

□

G Expo-power Utilities

We illustrate the expo-power utility function, a two-parameter family proposed by Saha (1993) to capture increasing/decreasing absolute/relative risk aversion,[26] suffers from the same problems as CARA and CRRA utilities. The increasing and concave expo-power utility function is given by

$$u_{a,r}(x) = \frac{1 - \mathrm{e}^{-ax^{1-r}}}{a},$$

for $x \geq 0$ and positive a and $0 \leq r < 1$. Note that the absolute risk aversion is given by

$$A_{a,r}(x) = \frac{r}{x} + a(1-r)x^{-r},$$

for $x > 0$, which is decreasing, constant, or increasing depending on the parameters. Thus, unlike CARA or CRRA utility, the family of expo-power utilities, parameterized by a and r, is not totally ordered by absolute/relative risk aversion.

Importantly, for any pairs of coefficients (a_1, r_1) and (a_2, r_2) such that $A_{a_1,r_1}(x) \geq A_{a_2,r_2}(x)$ so that $U_1 = \mathbb{E}[u_{a_1,r_1}(X)]$ is more risk-averse than $U_2 = \mathbb{E}[u_{a_2,r_2}(X)]$, it must hold that $r_1 = r_2$ and $a_1 > a_2$. A simple calculation shows that when $r_1 = r_2$ and $a_1 > a_2$, $u''_{a_2,r_2}(x)/u''_{a_1,r_1}(x)$ tends to infinity as $x \to \infty$. Thus, we have the following corollary.

Corollary 3. *For any continuous noise assignments Φ^A, Φ^B and distinct expo-power utility functions U_A,U_B such that U_A is more risk-averse than U_B, there are reversals.*

Corollary 3 highlights that the paradoxical properties arising from CARA (CRRA) utilities are pervasive, and that they do not depend on the strong assumption of constant absolute (relative) risk aversion.

H Axiomatic Foundation of SSEU

In this section, we formulate axioms on stochastic choice rules that underpin SSEU models. Formally, a stochastic choice rule $\rho : \mathcal{L} \times \mathcal{L} \to [0, 1]$ is any function satisfying

$$\rho(X, Y) + \rho(Y, X) = 1,$$

[26]Notably, Holt and Laury (2002) use the expo-power utility function to model increasing relative risk aversion exhibited in their data.

where $\rho(X, Y)$ is interpreted as the probability that X is chosen over Y. We say that ρ is monotone in a utility $U\colon \mathcal{L} \to \mathbb{R}$ if for all X, Y, Z, $U(X) \geq U(Y)$ implies that $\rho(X, Z) \geq \rho(Y, Z)$. As the following lemma states, ρ must be a function of utilities.

Lemma 6. *If ρ is monotone in a utility U, then there is a function $H\colon \mathbb{R}^2 \to [0, 1]$ such that for all $X, Y \in \mathcal{L}$,*

$$\rho(X, Y) = H(U(X), U(Y)).$$

Moreover, H is increasing in the first argument, decreasing in the second argument, and satisfies $H(s, t) + H(t, s) = 1$.

Proof of Lemma 6. For all $s, t \in \mathrm{Im}(U)$, define $\tilde{H}(s, t) = \rho(X, Y)$ for some X and Y with $U(X) = s, U(Y) = t$. To see that $\tilde{H}$ is well-defined, suppose that $U(X') = s, U(Y') = t$. Then, since ρ is monotone in U,

$$\rho(X', Y') = \rho(X, Y') = 1 - \rho(Y', X) = 1 - \rho(Y, X) = \rho(X, Y).$$

Clearly,

$$\tilde{H}(s, t) + \tilde{H}(t, s) = \rho(X, Y) + \rho(Y, X) = 1.$$

Moreover, for all $s_1, s_2, t_1, t_2 \in \mathrm{Im}(U)$, if $s_1 \leq s_2, t_1 \geq t_2$ then

$$\tilde{H}(s_1, t_1) \leq \tilde{H}(s_2, t_1) = 1 - \tilde{H}(t_1, s_2) \leq 1 - \tilde{H}(t_2, s_2) = \tilde{H}(s_2, t_2).$$

Define the lower and upper extensions to $\mathbb{R}^2$:

$$H_L(x, y) = \sup\{\tilde{H}(s, t) \mid s, t \in \mathrm{Im}(U), s \leq x, t \geq y\}, \quad \text{setting} \quad \sup \emptyset = 0,$$
$$H_U(x, y) = \inf\{\tilde{H}(s, t) \mid s, t \in \mathrm{Im}(U), s \geq x, t \leq y\}, \quad \text{setting} \quad \inf \emptyset = 1.$$

Note that $H_L \leq H_U$. Define $H(x, y) = \frac{1}{2}(H_L(x, y) + H_U(x, y))$. Clearly, H extends $\tilde{H}$, since if $x, y \in \mathrm{Im}(U)$, then $H_L(x, y) = H_U(x, y) = \tilde{H}(x, y)$.

Moreover, H_L and H_U are monotone in their first argument; hence, so is their average, H. Finally, note that

$$\begin{aligned}
H_L(y, x) &= \sup\{\tilde{H}(s, t) \mid s, t \in \mathrm{Im}(U), s \leq y, t \geq x\} \\
&= \sup\{1 - \tilde{H}(t, s) \mid s, t \in \mathrm{Im}(U), s \leq y, t \geq x\} \\
&= 1 - \inf\{\tilde{H}(t, s) \mid t, s \in \mathrm{Im}(U), t \geq x, s \leq y\} \\
&= 1 - H_U(x, y).
\end{aligned}$$

Hence, $H(x, y) + H(y, x) = 1$, and it follows that H is decreasing in its second argument. ☐

Since monotonicity in utility is an ordinal requirement, we could rather impose that ρ is monotone in a weak order $\succsim$ on $\mathcal{L}$, i.e., if $X \succsim Y$ then $\rho(X, Z) \geq \rho(Y, Z)$ for all Z.[27] If we additionally impose that $X \sim Y$ whenever X and Y have the same distribution and that $\succsim$ satisfy the expected-utility axioms: independence and continuity, then the conclusion of Lemma 6 holds for some vNM utility U.

I Sum-ex and Sum-power Utility Functions

In Section 6.1, we demonstrated that for $h > \ell \geq 0$ the utility families

$$u_a(x; h, \ell) = a \cdot \text{cara}_h(x) + \text{cara}_\ell(x) \qquad a \geq 0$$

and

$$v_a(x; h, \ell) = a \cdot \text{crra}_h(x) + \text{crra}_\ell(x) \qquad a \geq 0,$$

which are parametrized by the coefficient a placed on the more risk-averse cara or crra utility do not lead to reversals under homoskedastic Fechnerian noise. A utility function taking the form $u_a(x; h, \ell)$ is a sum-ex utility and one taking the form $v_a(x; h, \ell)$ is a sum-power utility. In this appendix, we briefly discuss the axioms that characterize the sum-ex and sum-power families as well as an interpretation of the risk parameter a.

I.1 One-switch Utilities

Bell (1988, 1996) showed that sum-ex and sum-power utilities are characterized by *one-switch* properties on preferences over gambles and decreasing absolute risk aversion.

Definition 2. *We say that a Bernoulli utility u satisfies the* additive one-switch rule *if, for all X and Y*

$$\mathbb{E}[u(X + t) - u(Y + t)]$$

is single-crossing in t. Likewise, we say that u satisfies the multiplicative one-switch rule *if, for all X and Y*

$$\mathbb{E}[u(kX) - u(kY)]$$

is single-crossing in $k > 0$.

[27] For monotonicity to hold, the weak order $\succsim$ must be defined in terms of ρ, with $X \succsim Y$ whenever $\rho(X, Y) \geq 1/2$.

The family of CARA utilities is similarly characterized by a zero-switch condition, which requires that preferences over gambles are invariant to background wealth. However, we often observe individuals who become less risk averse as their wealth levels increase, exhibiting *decreasing absolute risk aversion* (DARA). Formally, we say that a Bernoulli utility function u exhibits DARA if $-u''(x)/u'(x)$ is decreasing in x.

As argued by Bell (1988), the additive one-switch property is normatively appealing under DARA, since one may start to prefer a riskier gamble once their wealth exceeds some point and their aversion to risk diminishes. However, this preference should not be reverted at even higher levels of background wealth, at which the individual is even more risk tolerant. Similarly, Bell (1996) argues that the multiplicative one-switch rule is a normatively appealing criterion for an individual comparing two investments with different stochastic returns.

The following proposition is due to Bell (1988, 1996) and Pedersen and Satchell (2001).

Proposition 7. *Suppose a concave infinitely differentiable Bernoulli utility function u satisfies decreasing absolute risk aversion. Then*

1. *u satisfies the additive one-switch rule if and only if $u(x)$ is a sum-ex utility function;*
2. *u satisfies the multiplicative one-switch rule if and only if $u(x)$ is a sum-power utility function.*

I.2 Interpreting the Risk Coefficient

In Section 6.1, we showed that for sum-ex and sum-power utilities

$$u_a(x; h, \ell) = a \cdot \text{cara}_h(x) + \text{cara}_\ell(x), \quad v_a(x; h, \ell) = a \cdot \text{crra}_h(x) + \text{crra}_\ell(x),$$

a higher coefficient a corresponds to greater risk aversion in the Arrow-Pratt sense and an increased likelihood of choosing safer lotteries under identical noise. Thus, we advocated for replacing CARA and CRRA utilities with sum-ex and sum-power utilities in estimating risk attitudes.

In this section, we further provide an interpretation of the risk coefficient a that is to be estimated for these families. Specifically, we show that the coefficient a can be interpreted as a background wealth level. That is, the preference of an individual with a lower coefficient a is identical to that of an individual with a larger coefficient and more background wealth. This equivalence is formalized in the next proposition.

Proposition 8. *Let $a > b > 0$ and $h > \ell > 0$. Then there are $\alpha, \beta, t, k > 0$, $\gamma, \delta \in \mathbb{R}$ such that*

$$u_b(x; h, \ell) = \alpha \cdot u_a(x + t; h, \ell) + \gamma \qquad \text{for all } x \in \mathbb{R} \tag{15}$$

$$v_b(x; h, \ell) = \beta \cdot v_a(k \cdot x; h, \ell) + \delta \qquad \text{for all } x > 0. \tag{16}$$

Proof of Proposition 8. For sum-ex utility, let

$$\alpha = \left(\frac{a}{b}\right)^{\frac{\ell}{h-\ell}}, \qquad \gamma = \frac{b - \alpha \cdot a}{h} + \frac{1 - \alpha}{\ell}, \qquad t = \frac{1}{h - \ell} \ln\left(\frac{a}{b}\right).$$

For sum-power utility, if $\ell \neq 1$ and $h \neq 1$ let

$$\beta = \left(\frac{a}{b}\right)^{\frac{\ell-1}{h-\ell}}, \qquad \delta = \frac{a \cdot \beta - b}{1 - h} + \frac{\beta - 1}{1 - \ell}, \qquad k = \beta^{\frac{1}{\ell-1}}.$$

Plugging in these parameters, we obtain (15) and (16). For h or ℓ equal to 1, taking the limits of the expressions for β, δ, and γ works as well, completing the proof. □

Proposition 8 highlights a benefit that sum-ex and sum-power share with CARA and CRRA in the deterministic setting. One important feature of CARA utility is that background wealth does not affect preferences and thus does not need to be separately estimated. By Proposition 8, the estimation of the coefficient on the higher CARA utility can itself be interpreted as an estimation of background wealth which does not, therefore, need to be separately estimated. Likewise, CRRA is often used as a preference specification over investments where it has the benefit that the level of stakes to be invested do not need to be estimated since they do not affect the preference. While sum-power preferences are not invariant to changes in stakes, the coefficient on the higher CRRA utility can be interpreted as revealing the level of stakes to be invested, which do not then need to be separately estimated.